Politechnika Wrocławska

Plastyczna analiza MES panelu trójwarstwowego z zastosowaniem funkcjonału Hu-Washizu

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Plan prezentacji

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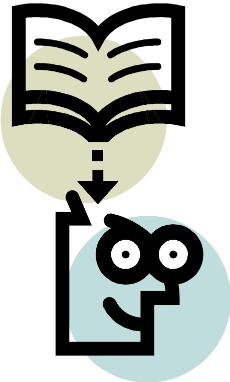
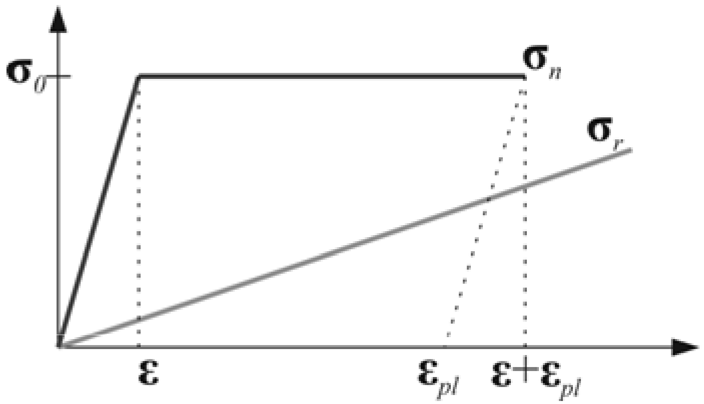
3. Zmodyfikowany funkcjonał Hu-Washizu
4. Równania MES
5. Algorytm rozwiązania
6. Przykłady



Model panelu



$$\sigma_n = D_n \varepsilon$$
$$\sigma_r = D_r\,(\varepsilon+\varepsilon_{pl})$$

- płaski stan naprężenia
- rdzeń: materiał sprężysty
- nakładki: kompozyt sprężysto-plastyczny; kryterium zniszczenia Tsai-Wu:

$$F_x = \frac{1}{R_{xt}} - \frac{1}{R_{xc}}, F_y = \frac{1}{R_{yt}} - \frac{1}{R_{yc}}$$

$$F_x\sigma_x + F_y\sigma_y + F_{xx}\sigma_x^{\,2} + F_{yy}\sigma_y^{\,2} + 2F_{xy}\sigma_x\sigma_y + F_t\tau_{xy}^{\,2} - 1 = 0 \qquad F_{xx} = \frac{1}{R_{xt}R_{xc}}, F_{yy} = \frac{1}{R_{yt}R_{yc}}$$

$$F_t = \frac{1}{R_s^{\,2}}, F_{xy} = -\frac{1}{2}\sqrt{F_{xx}F_{yy}}$$

Model panelu: kryterium zniszczenia

Kryterium Tsai-Wu w notacji macierzowej:

$$\mathbf{F}_m = \begin{bmatrix} F_{xx} & F_{xy} & 0 \\ F_{xy} & F_{yy} & 0 \\ 0 & 0 & F_t \end{bmatrix}$$

$$\mathbf{F}_v{}^T \boldsymbol{\sigma}_n + \boldsymbol{\sigma}_n{}^T \mathbf{F}_m \boldsymbol{\sigma}_n - 1 = 0$$

$$\mathbf{F}_v = \begin{Bmatrix} F_x \\ F_y \\ 0 \end{Bmatrix}, \boldsymbol{\sigma}_n = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

Oznaczając przez **N** siły przekrojowe panelu, po przekształceniach otrzymano kryterium zniszczenia **panelu**:

$$\boxed{\Phi(\mathbf{N}) = \mathbf{N}^T \mathbf{F}_{m2} \mathbf{N} + (\mathbf{F}_{v2}{}^T - 2\boldsymbol{\varepsilon}_{pl}{}^T \mathbf{F}_{m3})\mathbf{N} + \boldsymbol{\varepsilon}_{pl}{}^T \mathbf{F}_{m4} \boldsymbol{\varepsilon}_{pl} - \mathbf{F}_{v3}{}^T \boldsymbol{\varepsilon}_{pl} - 1 = 0}$$

$$\mathbf{F}_{m2} = \mathbf{D}^{-1}\mathbf{D}_n \mathbf{F}_m \mathbf{D}_n \mathbf{D}^{-1}$$
$$\mathbf{F}_{m3} = \mathbf{D}_r h_r \mathbf{D}^{-1}\mathbf{D}_n \mathbf{F}_m \mathbf{D}_n \mathbf{D}^{-1}$$
$$\mathbf{F}_{m4} = \mathbf{D}_r h_r \mathbf{D}^{-1}\mathbf{D}_n \mathbf{F}_m \mathbf{D}_n \mathbf{D}^{-1}\mathbf{D}_r h_r$$
$$\mathbf{F}_{v2}{}^T = \mathbf{F}_v{}^T \mathbf{D}_n \mathbf{D}^{-1},\ \mathbf{F}_{v3}{}^T = \mathbf{F}_v{}^T \mathbf{D}_n \mathbf{D}^{-1}\mathbf{D}_r h_r$$

gdzie sztywność panelu:

$$\mathbf{D} = 2\mathbf{D}_n h_n + \mathbf{D}_r h_r$$

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Model panelu: kryterium zniszczenia

zmiana $\boldsymbol{\varepsilon}_{pl}$ nakładek → zmiana kryterium zniszczenia panelu:

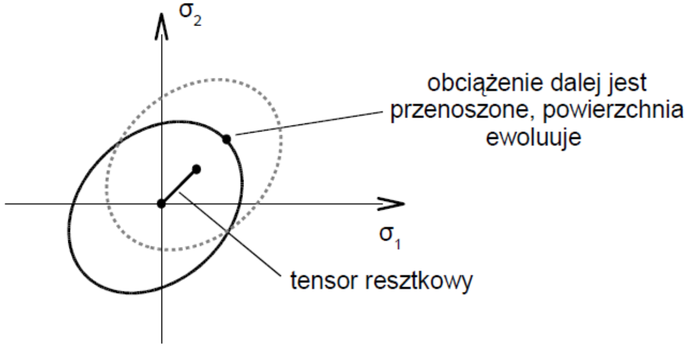
$$\Phi(\mathbf{N};\boldsymbol{\varepsilon}_{pl}+\Delta\boldsymbol{\varepsilon}_{pl}) = \mathbf{N}^T\mathbf{F}_{m2}\mathbf{N} + (\mathbf{F}_{v2}{}^T - 2(\boldsymbol{\varepsilon}_{pl}+\Delta\boldsymbol{\varepsilon}_{pl})^T\mathbf{D}_r h_r \mathbf{F}_{m2})\mathbf{N} + (\boldsymbol{\varepsilon}_{pl}+\Delta\boldsymbol{\varepsilon}_{pl})^T\mathbf{D}_r h_r \mathbf{F}_{m2}\mathbf{D}_r h_r(\boldsymbol{\varepsilon}_{pl}+\Delta\boldsymbol{\varepsilon}_{pl})$$
$$-\mathbf{F}_{v2}{}^T\mathbf{D}_r h_r(\boldsymbol{\varepsilon}_{pl}+\Delta\boldsymbol{\varepsilon}_{pl}) - 1 = 0$$

Podstawiając $\mathbf{D}_r h_r \Delta\boldsymbol{\varepsilon}_{pl} = \Delta\mathbf{N}$, po odpowiednich przekształceniach:

$$\Phi(\mathbf{N};\boldsymbol{\varepsilon}_{pl}+\Delta\boldsymbol{\varepsilon}_{pl}) = (\mathbf{N}-\Delta\mathbf{N})^T\mathbf{F}_{m2}(\mathbf{N}-\Delta\mathbf{N}) + \mathbf{F}_{v2}{}^T(\mathbf{N}-\Delta\mathbf{N}) - 2\boldsymbol{\varepsilon}_{pl}{}^T\mathbf{D}_r h_r\mathbf{F}_{m2}(\mathbf{N}-\Delta\mathbf{N}) + \boldsymbol{\varepsilon}_{pl}{}^T\mathbf{D}_r h_r\mathbf{F}_{m2}\mathbf{D}_r h_r\boldsymbol{\varepsilon}_{pl}$$
$$-\mathbf{F}_{v2}{}^T\mathbf{D}_r h_r\boldsymbol{\varepsilon}_{pl} - 1 = \boxed{\Phi(\mathbf{N}-\Delta\mathbf{N};\boldsymbol{\varepsilon}_{pl})}$$

wzmocnienie kinematyczne:

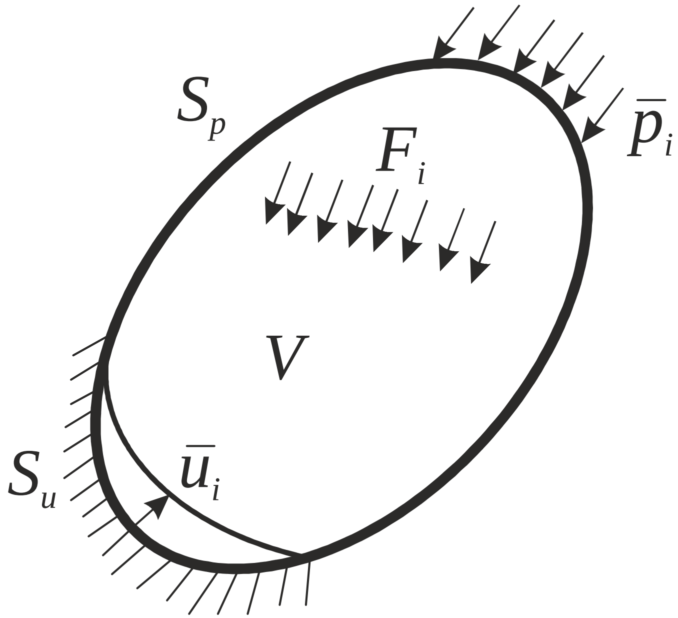
$\Delta\mathbf{N}$ - tensor resztkowy



Łodygowski T. ,Metody Komputerowe II, Alma Mater, 2004/2005

Funkcjonał Hu-Washizu - pierwotny

$$\Pi\left(\sigma_{ij},\varepsilon_{ij},u_i\right)=\frac{1}{2}\int_V C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}dV-\int_V \sigma_{ij}\left[\varepsilon_{ij}-\frac{1}{2}\left(u_{i},_{j}+u_{j},_{i}\right)\right]dV-\int_V F_i u_i dV-$$
$$-\int_{S_u}\sigma_{ij}n_j\left(u_i-\overline{u}_i\right)dS-\int_{S_p}\overline{p}_i u_i dS$$



$$\delta\Pi\left(\sigma_{ij},\varepsilon_{ij},u_i\right)=0\Rightarrow$$

$$\left.\begin{aligned}&\sigma_{ij},_{j}+F_i=0\\&\varepsilon_{ij}=\frac{1}{2}\left(u_{i},_{j}+u_{j},_{i}\right)\\&\sigma_{ij}=C_{ijkl}\varepsilon_{kl}\end{aligned}\right\}\ w\,V$$

$$u_i=\overline{u}_i \qquad na\ S_u$$
$$\sigma_{ij}n_j=\overline{p}_i \qquad na\ S_p$$

Przyrostowa forma funkcjonału

$$\Pi\left(\sigma_{ij}^{n+1},\varepsilon_{ij}^{n+1},u_i\right)=\frac{1}{2}\int_V C_{ijkl}\left(\varepsilon_{ij}^{n+1}-\varepsilon_{ij}^{n}\right)\left(\varepsilon_{kl}^{n+1}-\varepsilon_{kl}^{n}\right)dV-\int_V\left(\sigma_{ij}^{n+1}-\sigma_{ij}^{n}\right)\left[\varepsilon_{ij}^{n+1}-\frac{1}{2}\left(u_{i},_{j}+u_{j},_{i}\right)\right]dV$$

$$-\int_V \Delta F_i^{n+1}u_i dV+\ldots$$

$\sigma_{ij}^{n},\ \varepsilon_{ij}^{n}$ – naprężenie i odkształcenie w poprzednim kroku "n"

$\sigma_{ij}^{n+1},\ \varepsilon_{ij}^{n+1},\Delta F_i^{n+1}$ – naprężenie, odszktałcenie i przyrost obciążenia w aktualnym kroku "n+1"

$$\delta\Pi\left(\sigma_{ij}^{n+1},\varepsilon_{ij}^{n+1},u_i\right)=0\Rightarrow$$

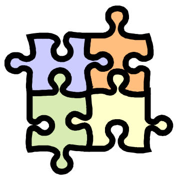
$$\left.\begin{aligned}&\left(\sigma_{ij}^{n+1}-\sigma_{ij}^{n}\right),_{j}+\Delta F_i^{n+1}=0\\&\varepsilon_{ij}^{n+1}=\frac{1}{2}\left(u_{i},_{j}+u_{j},_{i}\right)\\&\left(\sigma_{ij}^{n+1}-\sigma_{ij}^{n}\right)=C_{ijkl}\left(\varepsilon_{kl}^{n+1}-\varepsilon_{kl}^{n}\right)\end{aligned}\right\}\ w\,V$$

Modyfikacja związana z plastycznością

$$\Pi\left(\sigma_{ij}^{n+1},\varepsilon_{ij}^{n+1},u_i,\lambda\right)=\frac{1}{2}\int_V C_{ijkl}\left(\varepsilon_{ij}^{n+1}-\varepsilon_{ij}^{n}\right)\left(\varepsilon_{kl}^{n+1}-\varepsilon_{kl}^{n}\right)dV-\int_V\left(\sigma_{ij}^{n+1}-\sigma_{ij}^{n}\right)\left[\varepsilon_{ij}^{n+1}-\frac{1}{2}\left(u_{i,j}+u_{j,i}\right)\right]dV$$

$$-\int_V \Delta F_i^{n+1}u_i dV-\boxed{\int_V \lambda\Phi(\sigma_{ij}^{n+1})dV}+\ldots$$

$\Phi(\sigma_{ij}^{n+1})=0$ – warunek plastyczności



$$\delta\Pi\left(\sigma_{ij}^{n+1},\varepsilon_{ij}^{n+1},u_i\right)=0\Rightarrow$$

$$\left.\begin{aligned}
&\left(\sigma_{ij}^{n+1}-\sigma_{ij}^{n}\right)_{,j}+\Delta F_i^{n+1}=0\\
&\frac{1}{2}\left(u_{i,j}+u_{j,i}\right)=\varepsilon_{ij}^{n+1}+\boxed{\lambda\frac{\partial\Phi(\sigma_{ij}^{n+1})}{\partial\sigma_{ij}^{n+1}}}\\
&\left(\sigma_{ij}^{n+1}-\sigma_{ij}^{n}\right)=C_{ijkl}\left(\varepsilon_{kl}^{n+1}-\varepsilon_{kl}^{n}\right)\\
&\boxed{\Phi(\sigma_{ij}^{n+1})=0}
\end{aligned}\right\}\ w\,V$$

Postać funkcjonału w ujęciu MES

$$\Pi\left(\mathbf{N}^{n+1},\boldsymbol{\varepsilon}^{n+1},\mathbf{q},\lambda\right)=\frac{1}{2}\left(\boldsymbol{\varepsilon}^{n+1}-\boldsymbol{\varepsilon}^{n}\right)^{T}\mathbf{D}\left(\boldsymbol{\varepsilon}^{n+1}-\boldsymbol{\varepsilon}^{n}\right)V-\left(\Delta\mathbf{f}^{n+1}\right)^{T}\mathbf{q}-\left(\mathbf{N}^{n+1}-\mathbf{N}^{n}\right)^{T}\left(\boldsymbol{\varepsilon}^{n+1}-\mathbf{B}\mathbf{q}\right)V-\lambda\Phi V$$

V – miara ES

$\mathbf{B}$ – macierz geometryczna

$\mathbf{q}$ – wektor przemieszczeń węzłowych

$$\Phi=(\mathbf{N}^{n+1})^{T}\mathbf{F}_{m2}\mathbf{N}^{n+1}+(\mathbf{F}_{v2}{}^{T}-2\boldsymbol{\varepsilon}_{pl}{}^{T}\mathbf{F}_{m3})\mathbf{N}^{n+1}+\boldsymbol{\varepsilon}_{pl}{}^{T}\mathbf{F}_{m4}\boldsymbol{\varepsilon}_{pl}-\mathbf{F}_{v3}{}^{T}\boldsymbol{\varepsilon}_{pl}-1$$

$$\frac{\partial\Pi}{\partial\mathbf{q}^{n+1}}=0:\quad \mathbf{B}^{T}\left(\mathbf{N}^{n+1}-\mathbf{N}^{n}\right)V-\Delta\mathbf{f}^{n+1}=0$$

$$\frac{\partial\Pi}{\partial\boldsymbol{\varepsilon}^{n+1}}=0:\quad \mathbf{D}\left(\boldsymbol{\varepsilon}^{n+1}-\boldsymbol{\varepsilon}^{n}\right)V-\left(\mathbf{N}^{n+1}-\mathbf{N}^{n}\right)V=0$$

$$\frac{\partial\Pi}{\partial\mathbf{N}^{n+1}}=0:\quad -\left(\boldsymbol{\varepsilon}^{n+1}-\mathbf{B}\mathbf{q}\right)V-\boxed{\lambda(2\mathbf{F}_{m2}\mathbf{N}^{n+1}+\mathbf{F}_{v2}-2\mathbf{F}_{m3}{}^{T}\boldsymbol{\varepsilon}_{pl})}V=0$$

$$\frac{\partial\Pi}{\partial\lambda}=0:\quad -((\mathbf{N}^{n+1})^{T}\mathbf{F}_{m2}\mathbf{N}^{n+1}+(\mathbf{F}_{v2}{}^{T}-2\boldsymbol{\varepsilon}_{pl}{}^{T}\mathbf{F}_{m3})\mathbf{N}^{n+1}+\boldsymbol{\varepsilon}_{pl}{}^{T}\mathbf{F}_{m4}\boldsymbol{\varepsilon}_{pl}-\mathbf{F}_{v3}{}^{T}\boldsymbol{\varepsilon}_{pl}-1)V=0$$

Postać funkcjonału w ujęciu MES

$$\boldsymbol{\varepsilon}_{pl} = \lambda(2\mathbf{F}_{m2}\mathbf{N}^{n+1} + \mathbf{F}_{v2} - 2\mathbf{F}_{m3}{}^{T}\boldsymbol{\varepsilon}_{pl}) \;\rightarrow\; (\mathbf{I} + 2\lambda\mathbf{F}_{m3}{}^{T})\boldsymbol{\varepsilon}_{pl} = \lambda(2\mathbf{F}_{m2}\mathbf{N}^{n+1} + \mathbf{F}_{v2})$$

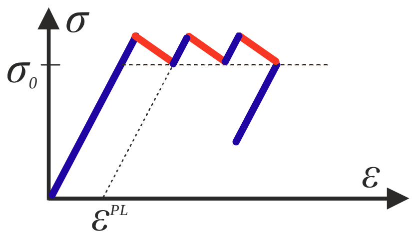
$$2\lambda\mathbf{F}_{m3}{}^{T} \ll \mathbf{I} \;\rightarrow\; \boldsymbol{\varepsilon}_{pl} \cong \lambda(2\mathbf{F}_{m2}\mathbf{N}^{n+1} + \mathbf{F}_{v2})$$

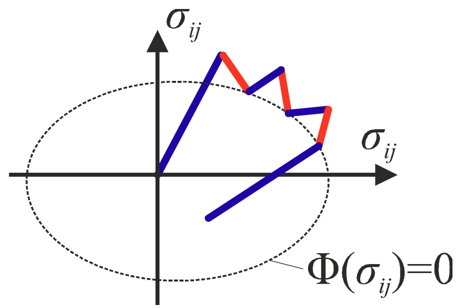
Po podstawieniu $\boldsymbol{\varepsilon}_{pl}$, równania można zapisać:

$$\boxed{\begin{bmatrix} 0 & 0 & \mathbf{B}^{T}V & 0 \\ 0 & \mathbf{D}V & -\mathbf{I}V & 0 \\ \mathbf{B}V & -\mathbf{I}V & 0 & \mathbf{K}_{S}^{\mathbf{N}\lambda} \\ 0 & 0 & \mathbf{K}_{S}^{\lambda\mathbf{N}} & \mathbf{K}_{S}^{\lambda\lambda} \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \boldsymbol{\varepsilon}^{n+1} \\ \mathbf{N}^{n+1} \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \Delta\mathbf{f}^{N+1} + \mathbf{B}^{T}\mathbf{N}^{n+1}V \\ \left(\mathbf{D}\boldsymbol{\varepsilon}^{n+1} - \mathbf{N}^{n+1}\right)V \\ 0 \\ -V \end{Bmatrix}}$$

$$\begin{aligned} \mathbf{K}_{S}^{\mathbf{N}\lambda} &= -(2\mathbf{F}_{m2}\mathbf{N}^{n+1} + \mathbf{F}_{v2} - 4\lambda\mathbf{F}_{m3}{}^{T}\mathbf{F}_{m2}\mathbf{N}^{n+1} - 2\lambda\mathbf{F}_{m3}{}^{T}\mathbf{F}_{v2})V \\ \mathbf{K}_{S}^{\lambda\mathbf{N}} &= -((\mathbf{N}^{n+1})^{T}\mathbf{F}_{m2} + \mathbf{F}_{v2}{}^{T} - 4\lambda(\mathbf{N}^{n+1})^{T}\mathbf{F}_{m2}\mathbf{F}_{m3} - 2\lambda\mathbf{F}_{v2}{}^{T}\mathbf{F}_{m3} \\ &\qquad + 4\lambda^{2}(\mathbf{N}^{n+1})^{T}\mathbf{F}_{m2}\mathbf{F}_{m4}\mathbf{F}_{m2} + 4\lambda^{2}\mathbf{F}_{v2}{}^{T}\mathbf{F}_{m4}\mathbf{F}_{m2} - 2\lambda\mathbf{F}_{v3}{}^{T}\mathbf{F}_{m2})V \\ \mathbf{K}_{S}^{\lambda\lambda} &= -(\lambda\mathbf{F}_{v2}{}^{T}\mathbf{F}_{m4}\mathbf{F}_{v2} - \mathbf{F}_{v3}{}^{T}\mathbf{F}_{v2})V \end{aligned}$$

Algorytm rozwiązania równań MES





Sprężystość lub odciążenie

$$\begin{bmatrix} 0 & 0 & \mathbf{B}^T V & 0 \\ 0 & \mathbf{D}V & -\mathbf{I}V & 0 \\ \mathbf{B}V & -\mathbf{I}V & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \boldsymbol{\varepsilon}^{n+1} \\ \mathbf{N}^{n+1} \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \Delta\mathbf{f}^{N+1} + \mathbf{B}^T\mathbf{N}^{n+1}V \\ \left(\mathbf{D}\boldsymbol{\varepsilon}^{n+1} - \mathbf{N}^{n+1}\right)V \\ \boldsymbol{\varepsilon}_{pl}V \\ 0 \end{Bmatrix}$$

Obciążenie plastyczne

$$\underbrace{\begin{bmatrix} 0 & 0 & \mathbf{B}^T V & 0 \\ 0 & \mathbf{D}V & -\mathbf{I}V & 0 \\ \mathbf{B}V & -\mathbf{I}V & 0 & \mathbf{K}_S^{\mathbf{N}\lambda} \\ 0 & 0 & \mathbf{K}_S^{\lambda\mathbf{N}} & \mathbf{K}_S^{\lambda\lambda} \end{bmatrix}}_{\mathbf{K}_S} \begin{Bmatrix} \mathbf{q} \\ \boldsymbol{\varepsilon}^{n+1} \\ \mathbf{N}^{n+1} \\ \lambda \end{Bmatrix} = \underbrace{\begin{Bmatrix} \Delta\mathbf{f}^{N+1} + \mathbf{B}^T\mathbf{N}^{n+1}V \\ \left(\mathbf{D}\boldsymbol{\varepsilon}^{n+1} - \mathbf{N}^{n+1}\right)V \\ 0 \\ -V \end{Bmatrix}}_{\mathbf{b}}$$

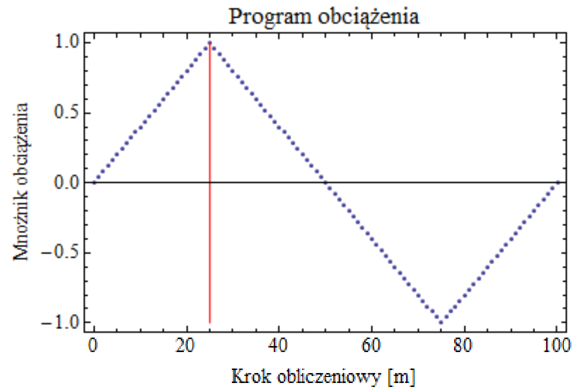
$$\mathbf{x} = \begin{Bmatrix} \mathbf{q} \\ \boldsymbol{\varepsilon}^{n+1} \\ \mathbf{N}^{n+1} \\ \lambda \end{Bmatrix}, \quad \mathbf{K}_T = \frac{\partial\left(\mathbf{K}_S\mathbf{x}\right)}{\partial\mathbf{x}} = \begin{bmatrix} 0 & 0 & \mathbf{B}^T V & 0 \\ 0 & \mathbf{D}V & -\mathbf{I}V & 0 \\ \mathbf{B}V & -\mathbf{I}V & \mathbf{K}_T^{\mathbf{NN}} & \mathbf{K}_T^{\mathbf{N}\lambda} \\ 0 & 0 & \mathbf{K}_T^{\lambda\mathbf{N}} & \mathbf{K}_T^{\lambda\lambda} \end{bmatrix}$$

$$\mathbf{x}_{i+1} = \mathbf{x}_i - \mathbf{K}_T^{-1}(\mathbf{x}_i)\left[\mathbf{K}_S(\mathbf{x}_i)\mathbf{x}_i - \mathbf{b}\right]$$

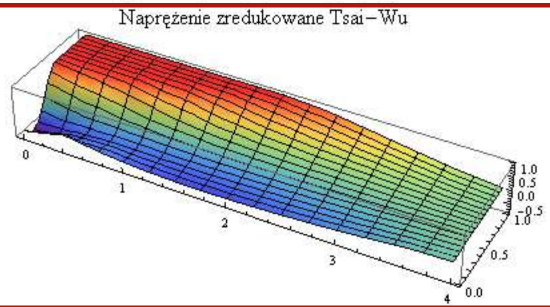
Przykłady: zginanie wspornika

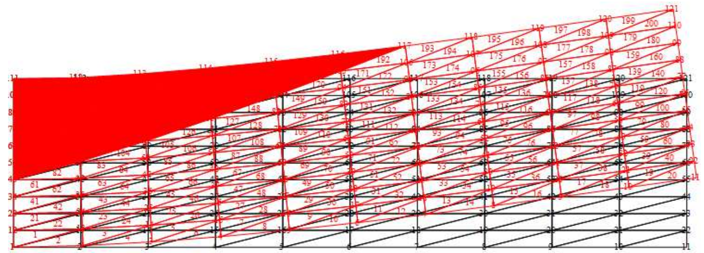
$$E_n = 200\ GPa$$
$$\nu_n = 0.3$$
$$h_n = 0.02\ m$$

$$E_r = 1\ GPa$$
$$\nu_r = 0.1$$
$$h_r = 0.06\ m$$



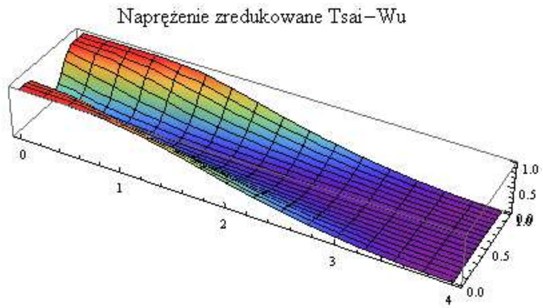
$$R_{xc} = R_{yc} = 50MPa$$
$$R_{xt} = R_{yt} = 200MPa$$
$$R_s = 100MPa$$

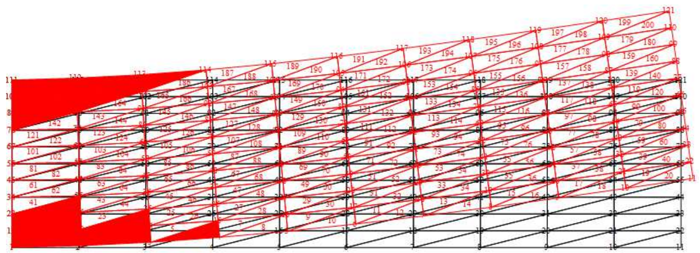




$$R_{xc} = R_{yc} = 85MPa$$
$$R_{xt} = R_{yt} = 85MPa$$
$$R_s = \frac{85}{\sqrt{3}}MPa$$

(Hubera-Misesa)





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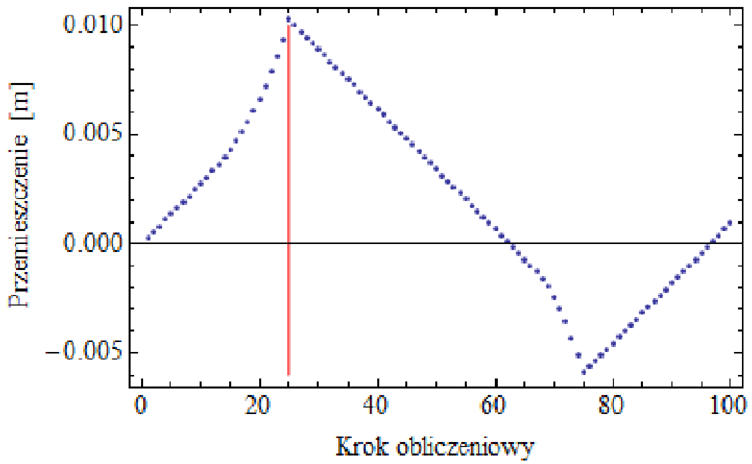
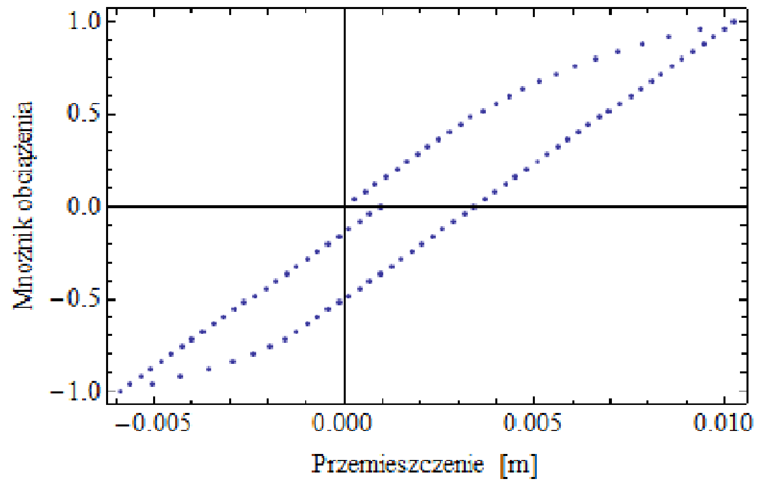
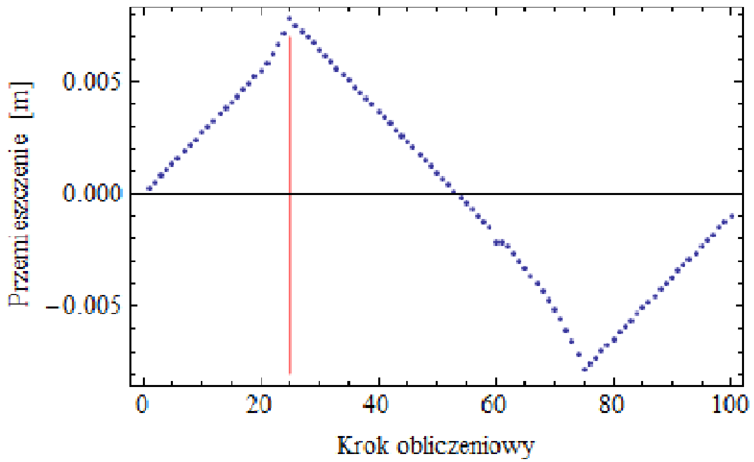
Przykłady: zginanie wspornika

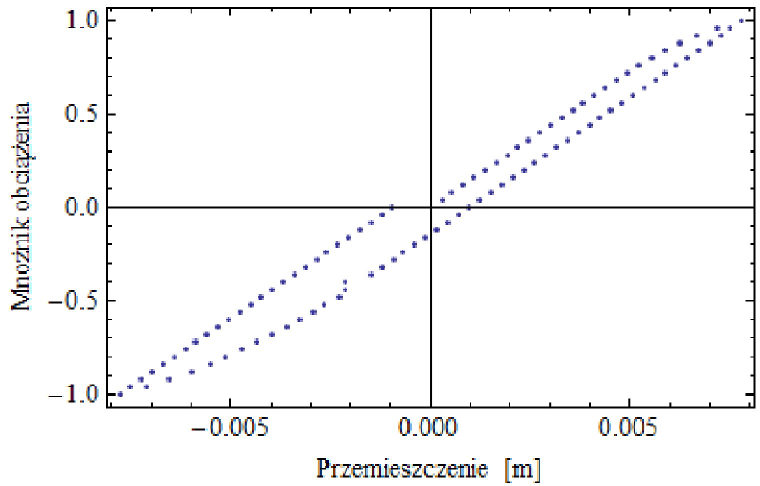
Tsai-Wu





Huber-Mises





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