Wrocław University of Technology

Modified Hu-Washizu principle as a general basis for FEM plasticity equations

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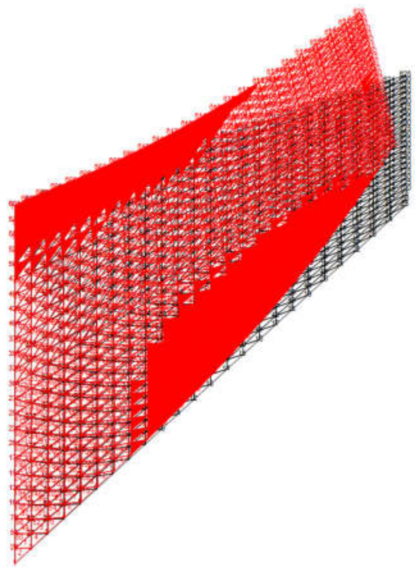
Plan

- Motivation
- Original and modified Hu-Washizu principle
- FEM equations
- FEM equations for HMH yield criterion
- FEM algorithm
- Verification and examples



Motivation

Universal principle for:

- any finite element (shape functions, number of dimensions)
- any yield function (with or without hardening)
- complete set of differential equations
  - equilibrium equations
  - Hook's law
  - geometrical relations
  - yield surface and flow rule equation

Definition → **Principle** → **Get direct FEM equations**

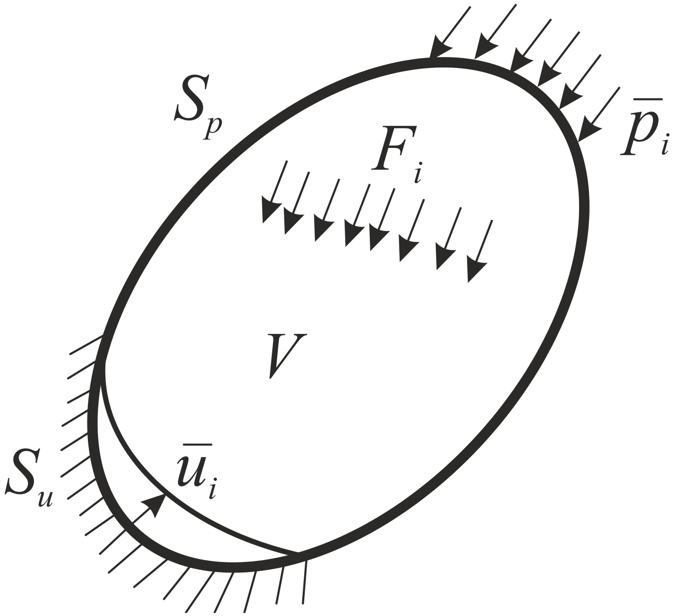
Definition
- FE
- Material
- Yield function

Principle
- Insert defined functions into the principle

Get direct FEM equations
- Solve equations with proposed simple algorithm

Original Hu-Washizu principle

$$\Pi\left(\sigma_{ij},\varepsilon_{ij},u_i\right)=\frac{1}{2}\int_V C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}dV-\int_V \sigma_{ij}\left[\varepsilon_{ij}-\frac{1}{2}\left(u_{i,j}+u_{j,i}\right)\right]dV-\int_V F_i\,u_i dV+\ldots$$



C_{ijkl}	elasticity tensor
$\sigma_{ij},\varepsilon_{ij}$	stress and strain tensors
u_i	displacement vector
F_i	body forces vector

$$\delta\Pi=0 \quad\Rightarrow\quad \left.\begin{aligned} &\sigma_{ij,j}+F_i=0\\ &\sigma_{ij}=C_{ijkl}\varepsilon_{kl}\\ &\varepsilon_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right)\end{aligned}\right\}\ \textit{in volume } V$$

Modified Hu-Washizu principle

$$\Pi\left(\sigma_{ij},\varepsilon_{ij},u_i,\lambda\right)=\frac{1}{2}\int_V C_{ijkl}\left(\varepsilon_{ij}-\varepsilon_{ij}^n\right)\left(\varepsilon_{kl}-\varepsilon_{kl}^n\right)dV-\int_V\left(\sigma_{ij}-\sigma_{ij}^n\right)\left[\varepsilon_{ij}+\overline{\varepsilon}_{ij}^n-\frac{1}{2}\left(u_{i,j}+u_{j,i}\right)\right]dV$$

$$-\int_V \Delta F_i\, u_i dV \boxed{-\int_V \lambda\Phi(\sigma_{ij})dV}+\dots$$

C_{ijkl}	elasticity tensor
$\sigma_{ij},\varepsilon_{ij}$	stress and elastic strain tensors (current)
$\sigma_{ij}^n,\varepsilon_{ij}^n$	stress and elastic strain tensors (previous)
$\overline{\varepsilon}_{ij}^n$	plastic strain tensor (previous)
u_i	displacement vector
ΔF_i	increment of body forces vector
Φ	plasticity surface formula
λ	coefficient of plastic strain increment

$$\delta\Pi=0 \quad \Rightarrow \quad \left.\begin{aligned} &\left(\sigma_{ij}-\sigma_{ij}^n\right)_{,j}+\Delta F_i=0 \\ &\left(\sigma_{ij}-\sigma_{ij}^n\right)=C_{ijkl}\left(\varepsilon_{kl}-\varepsilon_{kl}^n\right) \\ &\varepsilon_{ij}+\boxed{\overline{\varepsilon}_{ij}^n+\lambda\frac{\partial\Phi(\sigma_{ij})}{\partial\sigma_{ij}}}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right) \\ &\Phi(\sigma_{ij})=0 \end{aligned}\right\} \text{in volume } V$$

FEM equations

$$\Pi(\boldsymbol{\sigma},\boldsymbol{\varepsilon},\mathbf{q},\lambda)=\frac{1}{2}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^n)^T\mathbf{C}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^n)V-(\Delta\mathbf{f})^T\mathbf{q}-(\boldsymbol{\sigma}-\boldsymbol{\sigma}^n)^T\left(\boldsymbol{\varepsilon}+\bar{\boldsymbol{\varepsilon}}^n-\mathbf{Bq}\right)V-\lambda\Phi V$$

$$\delta\Pi=0 \quad\Rightarrow\quad \begin{bmatrix} 0 & 0 & \mathbf{B}^TV & 0 \\ 0 & \mathbf{C}V & -\mathbf{I}V & 0 \\ \mathbf{B}V & -\mathbf{I}V & \mathbf{K}_S^{\sigma\sigma} & \mathbf{K}_S^{\sigma\lambda} \\ 0 & 0 & \mathbf{K}_S^{\lambda\sigma} & \mathbf{K}_S^{\lambda\lambda} \end{bmatrix}\begin{Bmatrix}\mathbf{q}\\ \boldsymbol{\varepsilon}\\ \boldsymbol{\sigma}\\ \lambda\end{Bmatrix}=\begin{Bmatrix}\Delta\mathbf{f}+\mathbf{B}^T\boldsymbol{\sigma}^nV\\ (\mathbf{C}\boldsymbol{\varepsilon}^n-\boldsymbol{\sigma}^n)V\\ \bar{\boldsymbol{\varepsilon}}^nV\\ b^\lambda\end{Bmatrix}$$

For all further particular cryteria and examples plane stress state is assumed:

$$\boldsymbol{\sigma}=\mathbf{C}\boldsymbol{\varepsilon},\qquad \boldsymbol{\sigma}=\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}\quad \boldsymbol{\varepsilon}=\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}\quad \mathbf{C}=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}$$

FEM equations for Huber-Mises-Hencky yield criterion

Ideal plasticity:

$$\Phi(\boldsymbol{\sigma}) = \frac{1}{2}\boldsymbol{\sigma}^T \boldsymbol{\psi}\boldsymbol{\sigma} - 2\sigma_0^2, \quad \boldsymbol{\psi} = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 12 \end{bmatrix} \Rightarrow \begin{cases} \mathbf{K}_S^{\boldsymbol{\sigma}\lambda} = -\boldsymbol{\psi}\boldsymbol{\sigma}V, \quad \mathbf{K}_S^{\lambda\boldsymbol{\sigma}} = -\frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{\psi}V, \quad \mathbf{K}_S^{\boldsymbol{\sigma}\boldsymbol{\sigma}} = \mathbf{K}_S^{\lambda\lambda} = 0 \\ b^\lambda = -2\sigma_0^2 V \end{cases}$$

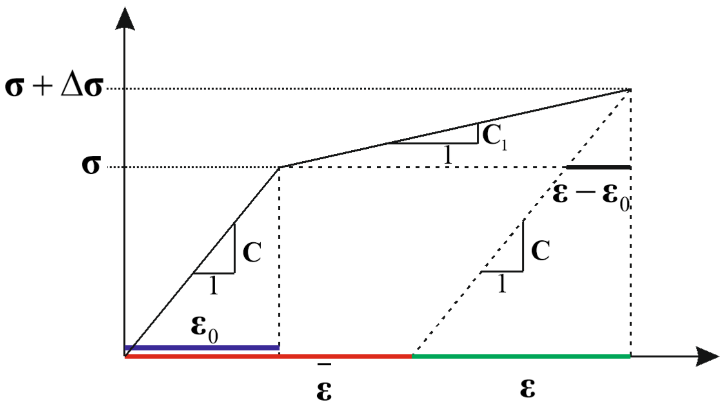
Kinematic hardening:

$\mathbf{C}_1 = \alpha\mathbf{C}, \; 0 < \alpha < 1$

$$\Phi(\boldsymbol{\sigma}) = \frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{\psi}\boldsymbol{\sigma} - \frac{\alpha}{1-\alpha}\boldsymbol{\sigma}^T\boldsymbol{\psi}\mathbf{C}\bar{\boldsymbol{\varepsilon}} + \frac{1}{2}\left(\frac{\alpha}{1-\alpha}\right)^2 \bar{\boldsymbol{\varepsilon}}^T\mathbf{C}\boldsymbol{\psi}\mathbf{C}\bar{\boldsymbol{\varepsilon}} - 2\sigma_0^2$$

$$\Downarrow$$

$$\begin{cases} \mathbf{K}_S^{\boldsymbol{\sigma}\boldsymbol{\sigma}} = 0, \; \mathbf{K}_S^{\boldsymbol{\sigma}\lambda} = -\left(\boldsymbol{\psi}\boldsymbol{\sigma} - \frac{\alpha}{1-\alpha}\boldsymbol{\psi}\mathbf{C}\left[\bar{\boldsymbol{\varepsilon}}^n + \Delta\bar{\boldsymbol{\varepsilon}}\right]\right)V \\ \mathbf{K}_S^{\lambda\boldsymbol{\sigma}} = -V\frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{\psi} + V\frac{\alpha}{1-\alpha}\left[\bar{\boldsymbol{\varepsilon}}^n + \Delta\bar{\boldsymbol{\varepsilon}}\right]^T\mathbf{C}\boldsymbol{\psi} \\ \mathbf{K}_S^{\lambda\lambda} = -V\frac{1}{2}\left(\frac{\alpha}{1-\alpha}\right)^2\frac{\Delta\bar{\boldsymbol{\varepsilon}}^T}{\lambda}\mathbf{C}\boldsymbol{\psi}\mathbf{C}\Delta\bar{\boldsymbol{\varepsilon}} - V\left(\frac{\alpha}{1-\alpha}\right)^2\frac{\Delta\bar{\boldsymbol{\varepsilon}}^T}{\lambda}\mathbf{C}\boldsymbol{\psi}\mathbf{C}\bar{\boldsymbol{\varepsilon}}^n \\ \mathbf{b}_\lambda = V\frac{1}{2}\left(\frac{\alpha}{1-\alpha}\right)^2\bar{\boldsymbol{\varepsilon}}^{n\,T}\mathbf{C}\boldsymbol{\psi}\mathbf{C}\bar{\boldsymbol{\varepsilon}}^n - V2\sigma_0^2 \end{cases}$$



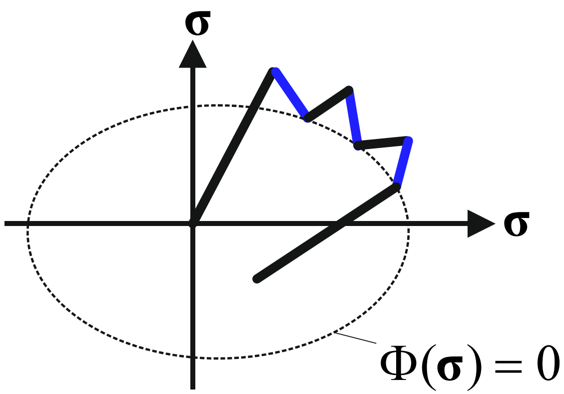
FEM algorithm

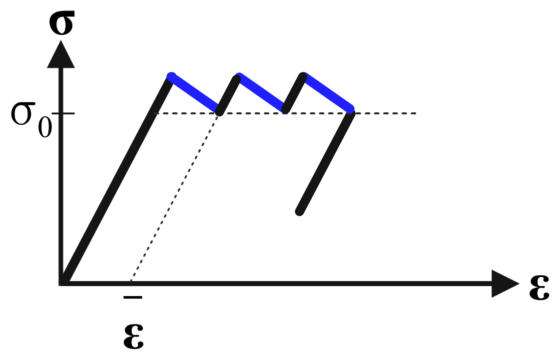
$$\underbrace{\begin{bmatrix} 0 & 0 & \mathbf{B}^T V & 0 \\ 0 & \mathbf{C}V & -\mathbf{I}V & 0 \\ \mathbf{B}V & -\mathbf{I}V & \mathbf{K}_S^{\boldsymbol{\sigma\sigma}} & \mathbf{K}_S^{\boldsymbol{\sigma}\lambda} \\ 0 & 0 & \mathbf{K}_S^{\lambda\boldsymbol{\sigma}} & \mathbf{K}_S^{\lambda\lambda} \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \boldsymbol{\varepsilon} \\ \boldsymbol{\sigma} \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \Delta\mathbf{f} + \mathbf{B}^T\boldsymbol{\sigma}^n V \\ \left(\mathbf{C}\boldsymbol{\varepsilon}^n - \boldsymbol{\sigma}^n\right)V \\ \bar{\boldsymbol{\varepsilon}}^n V \\ b^\lambda \end{Bmatrix}}_{\Downarrow \; \mathbf{K}_S\mathbf{x} = \mathbf{b}}$$

Newton method:

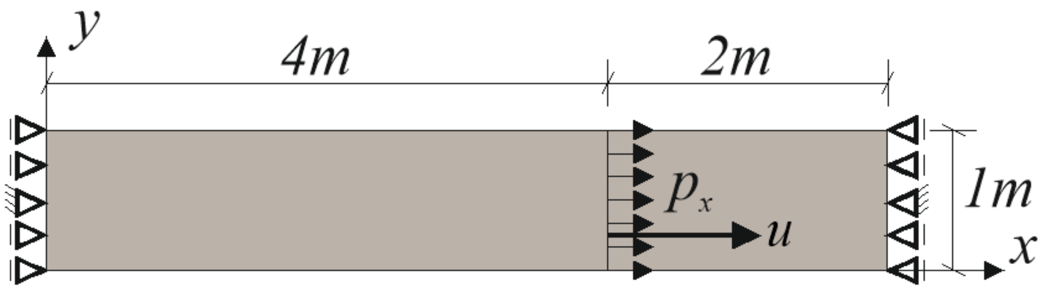
$$\mathbf{x}_{i+1} = \mathbf{x}_i - \mathbf{K}_T^{-1}(\mathbf{x}_i)\left[\mathbf{K}_S(\mathbf{x}_i)\mathbf{x}_i - \mathbf{b}\right]$$

$$\mathbf{K}_T = \frac{\partial\left(\mathbf{K}_S\mathbf{x}\right)}{\partial\mathbf{x}}$$

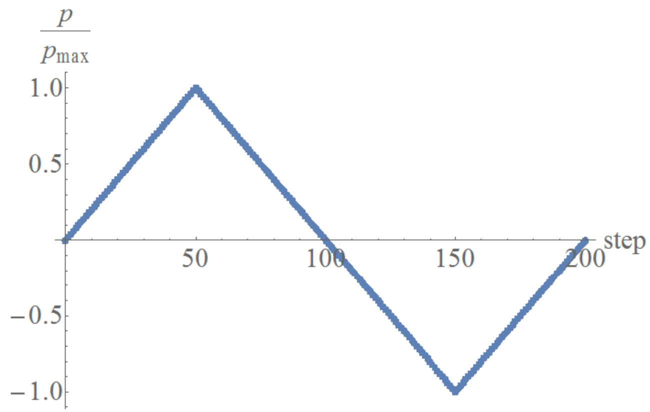


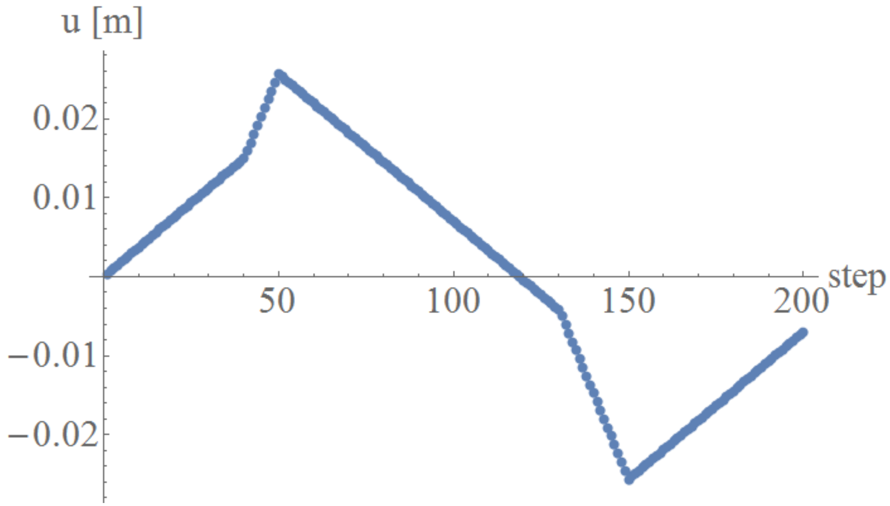


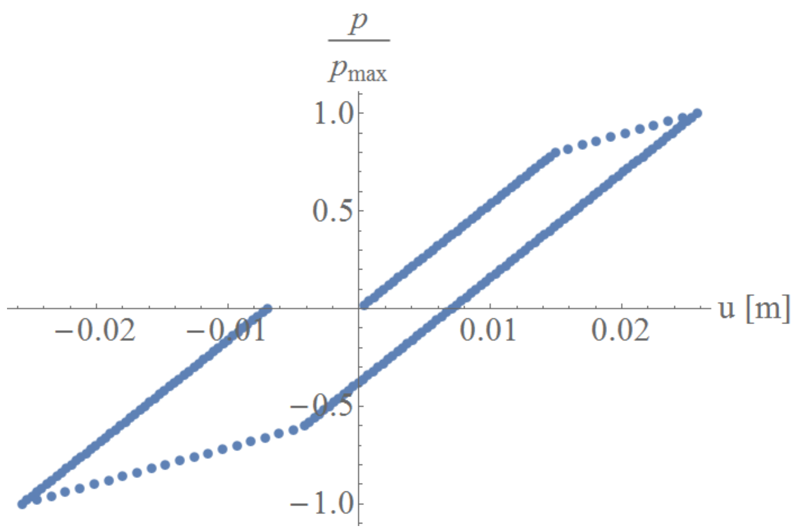
Verification – ideal plasticity



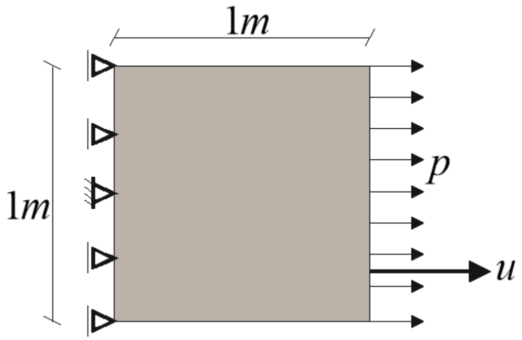
$E = 20MPa,\ \nu = 0,\ \sigma_0 = 150kPa,\ p_{max} = 280kPa,\ 96\ FEs$







Verification - kinematic hardening



$E = 20MPa$

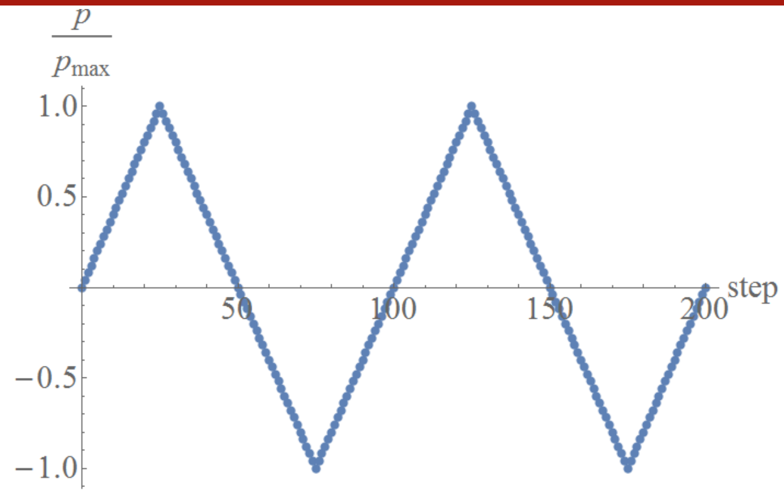
$\nu = 0.2$

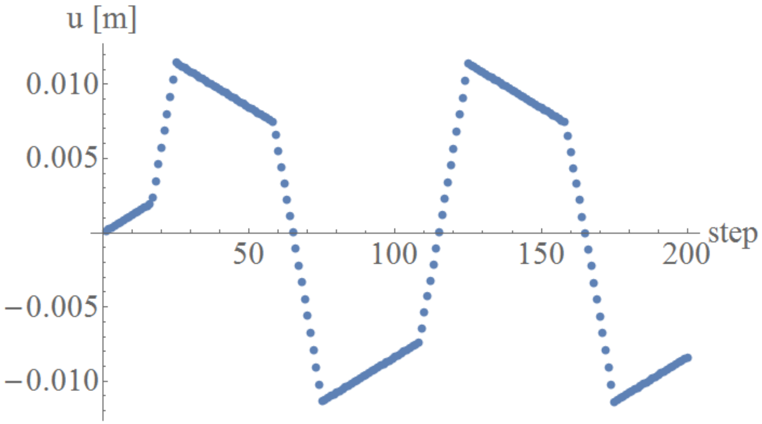
$\sigma_0 = 40kPa$

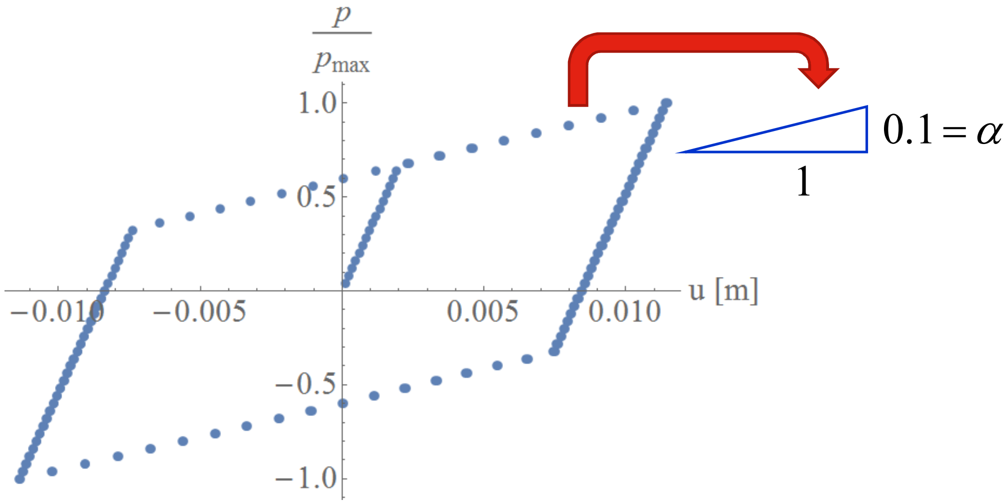
$\alpha = 0.1$

$p_{max} = 60kPa$

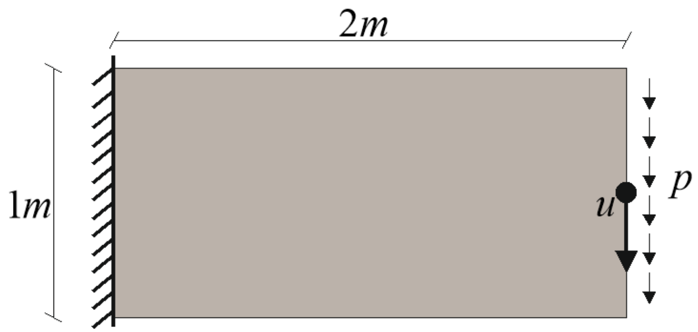
72 FEs







2D Cantilever beam



$E = 20MPa$

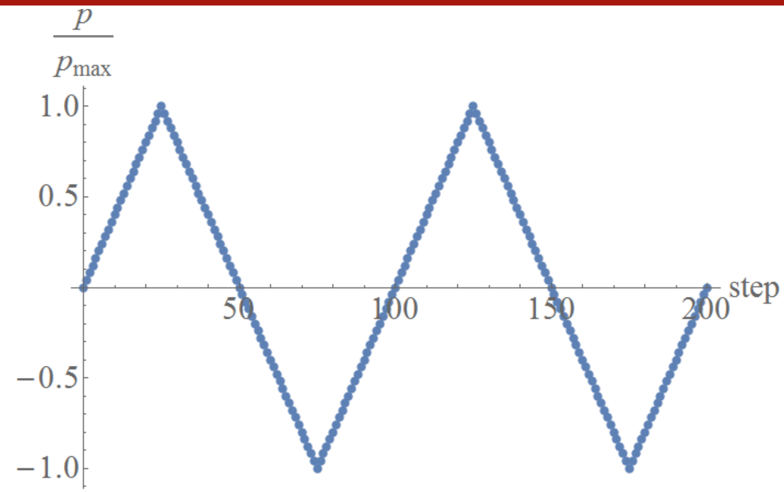
$\nu = 0.2$

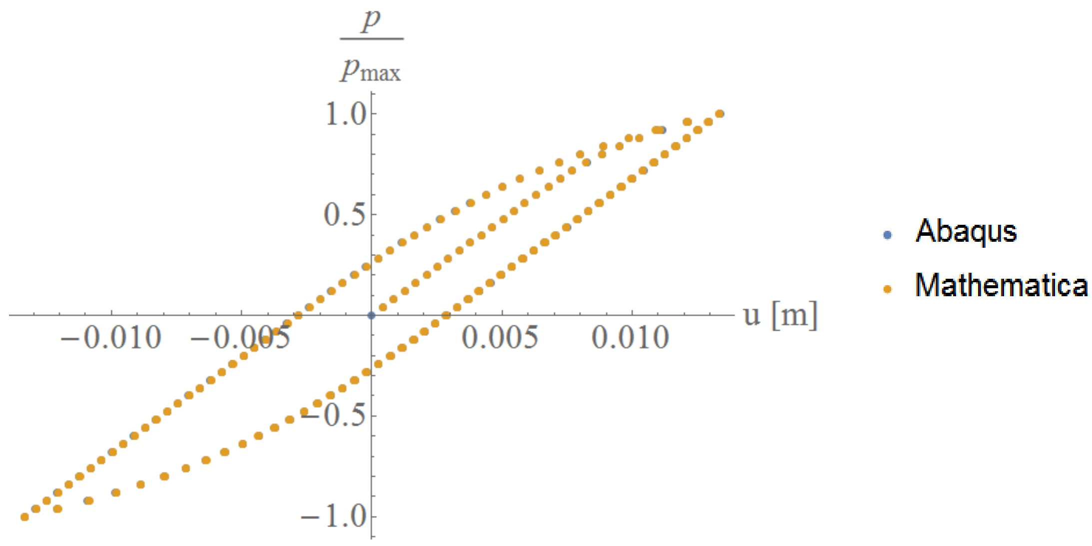
$\sigma_0 = 40kPa$

$\alpha = 0.1$

$p_{max} = 6kPa$

200 FEs





Literature

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